JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 19, No. 6, November–December 1996

Hankel Singular Values of Flexible Structures in Discrete Time

K. B. Lim*
NASA Langley Research Center, Hampton, Virginia 23681
and
W. Gawronski†
Jet Propulsion Laboratory, California Institute of Technology, Pasadena, California 91109

Approximate analytical expressions for controllability and observability grammian matrices and Hankel singular values of discrete linear time-invariant flexible structures are derived. The diagonal dominance property of the discrete grammians is shown, which results in the invariance of the principal directions. The approximate discrete Hankel singular values converge to the continuous formula with increased sampling rate, whereas the controllability and observability grammians go to zero and infinity, respectively. The approximate formula is accurate up to frequencies close to the Nyquist.

I. Introduction

It is well known that degrees of controllability and observability for linear systems are conveniently captured by the singular values of grammians. These singular values have a wide range of applications from system identification and model reduction to actuator and sensor placement for effective control and sensing configuration. Although the physical interpretation and approximating formula have been investigated in detail in the past for continuous systems (see, for example, Refs. 1–9), there is a significant lack of results for discrete systems, although the results are expected to be analogous to the continuous case. This need for results in the discrete domain is painfully clear, for example, when a control engineer is faced with the task of analysis and design of controllers for a large-order model of a discrete system.

In this paper, analytical expressions for controllability and observability grammian matrices and Hankel singular values (HSV) of discrete linear time-invariant flexible structures are derived. Results based on two types of models for discrete flexible structures are given: discretization of continuous systems via sampling and zero-order-hold (ZOH) and implicitly discrete models. The first type of model is typically obtained by analytical means, whereas the second type typically arises from system identification. The sampled/ZOH model is a parameterized model that allows direct comparisons to continuous singular values when the sampling rate is varied. Derivations of the approximate singular value formulas are given only for the first type of model, and the results based on the second type of parameterization are summarized as corollaries. For the class of flexible structures with small damping and distinct frequencies, the preceding formulas are significantly simplified. The approach is complementary to the earlier results on continuous time flexible structures reported in Refs. 5–7. Similar to the continuous case, the diagonal dominance property of the discrete grammians for small damping is shown. As a result, the approximate invariance of principal controllability and observability directions also holds for discrete time flexible structures. The dependence of the grammians on the sampling time and in particular their deviation from the corresponding continuous grammian is investigated. In particular, it is shown that the approximate discrete HSV formula converges to the approximate continuous formula with increased sampling rate, whereas the controllability and observability grammians go to zero and infinity, respectively. It is shown by numerical examples that the approximate formulas for singular values of discrete controllability and observability grammians and HSV are accurate up to frequencies close to the Nyquist frequency. Two levels of damping are assumed to evaluate the effect of violating the assumption of a lightly damped flexible structure.

II. Flexible Structure

A. Continuous Time

Let the triple (A, B, C) denote a modal state–space representation of a flexible structure with n structural modes. Following earlier definitions,[6–10] define the modal state vector x of dimension $n_2 \times 1$, where $n_2 = 2n$, such that

$$x = (\dot{\eta}_1 \quad \omega_1\eta_1 \quad \cdots \quad \dot{\eta}_n \quad \omega_n\eta_n)^T \tag{1}$$

Then the modal state equations take the form

$$\dot{x} = \mathrm{diag}(A_1, \ldots, A_n)x + \begin{bmatrix} B_{1*} \\ \vdots \\ B_{n*} \end{bmatrix} u \tag{2}$$

$$y = [C_{*1} \quad \cdots \quad C_{*n}]x \tag{3}$$

where

$$A_i = \begin{bmatrix} -2\zeta_i\omega_i & -\omega_i \\ \omega_i & 0 \end{bmatrix}, \qquad B_{i*} = \begin{bmatrix} b_i \\ 0 \end{bmatrix}$$
$$C_{*i} = [\, c_{ri} \quad (1/\omega_i)c_{di} \,] \tag{4}$$

and $i = 1, \ldots, n$, $b_i = \psi_i^T E$, $c_{di} = F\psi_i$, and $c_{ri} = G\psi_i$. Notice that for small damping

$$0 < \zeta_i \ll 1 \tag{5}$$

The preceding choice of the state vector gives the approximately normal state matrix and hence approximately orthogonal eigenvectors. For flexible structures with distinct natural frequencies, the steady-state controllability and observability grammians asymptotically (as $\zeta \to 0$) approach two-by-two block diagonal matrices as given in Refs. 6, 7, and 10,

$$\gamma_{ci}^2 = \frac{\beta_{ii}^2}{4\zeta_i\omega_i}, \qquad \gamma_{oi}^2 = \frac{\hat{\theta}_i^2}{4\zeta_i\omega_i}, \qquad \gamma_i^4 = \left(\frac{\beta_{ii}\hat{\theta}_i}{4\zeta_i\omega_i}\right)^2 \tag{6}$$

Received Nov. 17, 1995; revision received June 23, 1996; accepted for publication July 1, 1996; presented as Paper 96-3757 at the AIAA Guidance, Navigation, and Control Conference, San Diego, CA, July 29–31, 1996.

*Research Engineer, Guidance and Control Branch, Flight Dynamics and Control Division, MS 161.
†Technical Staff, Communications Ground Systems, MS 144-201.

where

$$\beta_{ij}^2 = b_i b_j^T \tag{7}$$

$$\hat{\theta}_i^2 = \left(1/\omega_i^2\right)c_{di}^T c_{di} + c_{ri}^T c_{ri} \tag{8}$$

are the modal grammian coefficients.[8,9]

B. Discrete Time

Two different forms of parameterizations of the discrete flexible structures are considered. The first form is used in the detailed derivations in the remaining sections, and the results based on the second form of parameterization are given as corollaries without details.

1. *Sampled/ZOH Model*

Consider a continuous flexible structure as defined by the block diagonal modal state-space representation in Sec. II.A and sampled at the outputs with period T and with a ZOH at the inputs. The state equation is given in this case by

$$x(k+1) = \tilde{A}x(k) + \tilde{B}u(k) \tag{9}$$

$$y(k) = \tilde{C}x(k) + \tilde{D}u(k) \tag{10}$$

where $\tilde{C} = C$ and $\tilde{D} = D$, whereas the discrete system matrices $\tilde{A}$ and $\tilde{B}$ are given by

$$\tilde{A} = e^{AT} = \text{blk-diag}[\tilde{A}_1(T), \ldots, \tilde{A}_n(T)] \tag{11}$$

$$\tilde{B} = \int_{kT}^{(k+1)T} \exp\{A[(k+1)T - \tau]\}\,d\tau B$$
$$= \text{blk-diag}(\tilde{M}_1, \ldots, \tilde{M}_n)B \tag{12}$$

where

$$\tilde{M}_i = \int_0^T \tilde{A}_i(\xi)\,d\xi$$

and by denoting the damped frequency of the continuous structure as $\omega_{di} = \omega_i\sqrt{(1-\zeta_i^2)}$, the ith block of $\tilde{A}$ is

$$\tilde{A}_i(T) = \frac{\exp(-\zeta_i\omega_i T)}{\omega_{di}}\begin{bmatrix} -\zeta_i\omega_i\sin(\omega_{di}T) + \omega_{di}\cos(\omega_{di}T) & -\omega_i\sin(\omega_{di}T) \\ \omega_i\sin(\omega_{di}T) & \zeta_i\omega_i\sin(\omega_{di}T) + \omega_{di}\cos(\omega_{di}T) \end{bmatrix} \tag{13}$$

2. *Implicitly Discrete Model*

In general, the state matrix of a discrete time model of a flexible structure may be fully populated. The following defines a similarity transformation to block diagonalize the state matrix.

Lemma 1: Let the quadruple (A_z, B_z, C_z, D_z) denote the discrete state-space matrices of a flexible structure. Let (z_i, v_i) denote the ith eigenvalue and eigenvector pair of A_z. The state transformation matrix

$$V = [\text{Re}(v_1) \quad -\text{Im}(v_1) \quad \cdots \quad \text{Re}(v_n) \quad -\text{Im}(v_n)] \tag{14}$$

block diagonalizes the state equations as in Eqs. (9) and (10), where

$$\tilde{A} = \text{blk-diag}[\tilde{A}_1(T), \ldots, \tilde{A}_n(T)] \tag{15}$$

$$\tilde{B} = V^{-1}B_z, \qquad \tilde{C} = C_z V, \qquad \tilde{D} = D_z \tag{16}$$

and

$$\tilde{A}_i = \begin{bmatrix} \text{Re}(z_i) & -\text{Im}(z_i) \\ \text{Im}(z_i) & \text{Re}(z_i) \end{bmatrix} \tag{17}$$

□

For a lightly damped flexible structure, its ith discrete eigenvalue lies just inside the unit circle and can be written as $z_i = \exp[(-\delta_i + j\psi_i)T]$, where $\delta_i > 0$. The $\tilde{A}_i$ matrix in Eq. (17) then becomes

$$\tilde{A}_i = \begin{bmatrix} \cos(\psi_i T) & -\sin(\psi_i T) \\ \sin(\psi_i T) & \cos(\psi_i T) \end{bmatrix} e^{-\delta_i T} \tag{18}$$

Since the preceding discrete eigenvalue is related to the eigenvalue of the corresponding sampled continuous signal, the following analogy holds: $\delta_i \leftrightarrow \zeta_i\omega_i$ and $\psi_i \leftrightarrow \omega_{di}$.

C. Small Damping Approximation

Assuming that the sampling rate is sufficiently fast such that the sampling theorem is satisfied (see, for example, page 111 in Ref. 11), i.e., $\omega_i \leq \pi/T$ for all i, one obtains from Eq. (5)

$$\zeta_i\omega_i T \ll 1 \tag{19}$$

The two-by-two block matrix $\tilde{A}_i$ in Eq. (13) can be approximated by

$$\tilde{A}_i(T) \cong \Psi_i(T)\exp(-\zeta_i\omega_i T) \tag{20}$$

where $\Psi_i(T)$ is an orthogonal matrix of the form

$$\Psi_i(T) = \begin{bmatrix} \cos(\omega_{di}T) & -\sin(\omega_{di}T) \\ \sin(\omega_{di}T) & \cos(\omega_{di}T) \end{bmatrix} \tag{21}$$

Note that Eqs. (20) and (21) are analogous to Eq. (18). Using Eq. (20), one can reduce the definite integral in Eq. (12) to

$$\tilde{B} \cong \text{blk-diag}(M_1, \ldots, M_n)B \tag{22}$$

where

$$M_i = \frac{1}{\omega_i^2}\begin{bmatrix} a_i & -b_i \\ b_i & a_i \end{bmatrix} \tag{23}$$

and

$$a_i = \omega_i\sin(\omega_i T) + \mathcal{O}(\zeta_i) \tag{24}$$

$$b_i = \omega_i[1 - \cos(\omega_i T)] + \mathcal{O}(\zeta_i) \tag{25}$$

III. Controllability Grammian

A. Definition

For the time interval (k_0T, k_1T), the discrete time controllability grammian $W_c(k_0, k_1)$ is defined in terms of the state transition matrix Φ and input matrix $\tilde{B}$,

$$W_c(k_0, k_1) = \sum_{k=k_0}^{k_1-1} \Phi(k_1, k+1)\tilde{B}\tilde{B}^T\Phi^T(k_1, k+1)$$
$$= P_c(k_1 - k_0)P_c^T(k_1 - k_0) \tag{26}$$

where $P_c(k_1 - k_0)$ is the discrete time controllability matrix

$$P_c(k_1 - k_0) = [\tilde{B} \quad \tilde{A}\tilde{B} \quad \cdots \quad \tilde{A}^{k_1-k_0-1}\tilde{B}] \tag{27}$$

It can be shown that the preceding grammian satisfies the following equation:

$$\tilde{A}W_c(k_0, k_1)\tilde{A}^T + \tilde{B}\tilde{B}^T = W_c(k_0, k_1) + \Phi(k_1, k_0)\tilde{B}\tilde{B}^T\Phi^T(k_1, k_0) \tag{28}$$

For asymptotically stable linear systems, the last term in Eq. (28) vanishes as $k_1 \to \infty$. This leads to the steady-state discrete time controllability grammian W_{c_∞}, which satisfies the following Sylvester equation:

$$\tilde{A}W_{c_\infty}\tilde{A}^T + \tilde{B}\tilde{B}^T = W_{c_\infty} \tag{29}$$

B. Closed-Form Solution

By taking advantage of the two-by-two block diagonal form of the state matrix in Eq. (11), Eq. (29) can be written as a set of two-by-two Sylvester equations,

$$\tilde{A}_i\left[W_{c\infty}\right]_{ij}\tilde{A}_j^T + [\tilde{B}\tilde{B}^T]_{ij} = \left[W_{c\infty}\right]_{ij} \tag{30}$$

where $i, j = 1, \ldots, n$, and

$$\tilde{A}_i = \tilde{A}_i(T) \tag{31}$$

$$[\tilde{B}\tilde{B}^T]_{ij} = \tilde{B}_i B_i B_j^T \tilde{B}_j \tag{32}$$

and $[W_{c\infty}]_{ij}$ is the (i, j)th two-by-two block of $[W_{c\infty}]$. For small damping, Eq. (30) can be approximated by

$$\exp(-\zeta_i\omega_i T)\Psi_i\left[W_{c\infty}\right]_{ij}\Psi_j^T\exp(-\zeta_j\omega_j T) - \left[W_{c\infty}\right]_{ij} = -[\tilde{B}\tilde{B}^T]_{ij} \tag{33}$$

and equivalently by postmultiplying by the orthogonal matrix Ψ_j, one obtains

$$\alpha_i\Psi_i\left[W_{c\infty}\right]_{ij} - \left[W_{c\infty}\right]_{ij}\Psi_j\alpha_j^{-1} = -[\tilde{B}\tilde{B}^T]_{ij}\Psi_j\alpha_j^{-1} \tag{34}$$

where

$$\alpha_i = \exp(-\zeta_i\omega_i T) \tag{35}$$

After some manipulation, it can be shown (see Appendix A) that the solution for the steady-state discrete time controllability grammian for flexible structures is given as follows.

Proposition 1:

$$\left[W_{c\infty}\right]_{ij} \cong -\frac{\beta_{ij}^2}{2\omega_i^2\omega_j^2}\operatorname{Re}\left(\Xi_1\frac{[Q_{ij}]_{11}}{\rho_{ij}} + \Xi_2\frac{[Q_{ij}]_{21}}{\mu_{ij}}\right) \tag{36}$$

where $i, j = 1, \ldots, n$, and

$$[Q_{ij}]_{11} = \lambda_j\{a_i a_j + b_i b_j + j(b_i a_j - a_i b_j)\} \tag{37}$$

$$[Q_{ij}]_{21} = \lambda_j\{-a_i a_j + b_i b_j + j(b_i a_j + a_i b_j)\} \tag{38}$$

$$\rho_{ij} = \alpha_i\lambda_i - \alpha_j^{-1}\lambda_j \tag{39}$$

$$\mu_{ij} = \alpha_i\lambda_i^* - \alpha_j^{-1}\lambda_j \tag{40}$$

$$\Xi_1 = \begin{bmatrix} 1 & j \\ -j & 1 \end{bmatrix}; \qquad \Xi_2 = \begin{bmatrix} -1 & -j \\ -j & 1 \end{bmatrix} \tag{41}$$

The asterisk denotes complex conjugate. □

For the state space parameterized as in Eqs. (15–17), the following results hold.

Corollary 1:

$$\left[W_{c\infty}\right]_{ij} = -\frac{1}{2\bar{\alpha}_j}\operatorname{Re}\left(\Xi_1\frac{[\bar{Q}_{ij}]_{11}}{\bar{\rho}_{ij}} + \Xi_2\frac{[\bar{Q}_{ij}]_{21}}{\bar{\mu}_{ij}}\right) \tag{42}$$

where $i, j = 1, \ldots, n$, and

$$[\bar{Q}_{ij}]_{11} = z_j\{a_{ij} + d_{ij} + j(c_{ij} - b_{ij})\} \tag{43}$$

$$[\bar{Q}_{ij}]_{21} = z_j\{-a_{ij} + d_{ij} + j(c_{ij} + b_{ij})\} \tag{44}$$

$$\bar{\rho}_{ij} = \bar{\alpha}_i z_i - \bar{\alpha}_j^{-1} z_j \tag{45}$$

$$\bar{\mu}_{ij} = \bar{\alpha}_i z_i^* - \bar{\alpha}_j^{-1} z_j \tag{46}$$

$$\bar{\alpha}_i = e^{-\delta_i T} \tag{47}$$

□

In the preceding corollary, z_i denotes the ith discrete eigenvalue, whereas a_{ij}, b_{ij}, c_{ij}, and d_{ij} are the input matrices defined by

$$[\tilde{B}\tilde{B}^T]_{ij} = \left[V^{-1}B_z B_z^T V^{-T}\right]_{ij} = \begin{bmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{bmatrix} \tag{48}$$

Note that Corollary 1 is an exact relationship.

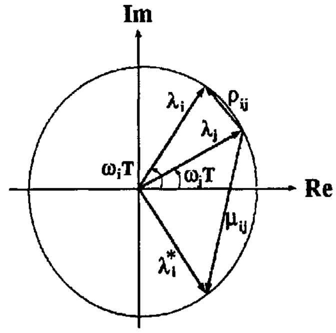


Fig. 1 Eigenvalues of flexible structure.

C. Diagonal Dominance

The denominator scalars in Eq. (36) can be expanded as follows:

$$\begin{bmatrix} \rho_{ij} \\ \mu_{ij} \end{bmatrix} = \begin{bmatrix} (1 - \zeta_i\omega_i T)\lambda_i - (1 + \zeta_j\omega_j T)\lambda_j + \mathcal{O}(\zeta_i^2) \\ (1 + \zeta_i\omega_i T)\lambda_i^* - (1 + \zeta_j\omega_j T)\lambda_j + \mathcal{O}(\zeta_i^2) \end{bmatrix} \tag{49}$$

where λ_i is the ith discrete eigenvalue of the ith two-by-two orthogonal matrix Ψ_i. For the off-diagonal block matrices where $i \neq j$,

$$\begin{bmatrix} \rho_{ij} \\ \mu_{ij} \end{bmatrix} = \begin{bmatrix} \lambda_i - \lambda_j - \zeta_i\omega_i T(\lambda_i + \lambda_j) + \mathcal{O}(\zeta_i^2) \\ \lambda_i^* - \lambda_j - \zeta_i\omega_i T(\lambda_i^* + \lambda_j) + \mathcal{O}(\zeta_i^2) \end{bmatrix} \cong \begin{bmatrix} \lambda_i - \lambda_j \\ \lambda_i^* - \lambda_j \end{bmatrix} \tag{50}$$

For the diagonal block matrices where $i = j$,

$$\begin{bmatrix} \rho_{ii} \\ \mu_{ii} \end{bmatrix} = \begin{bmatrix} -2\zeta_i\omega_i T\lambda_i + \mathcal{O}(\zeta_i^2) \\ \lambda_i^* - \lambda_i - \zeta_i\omega_i T(\lambda_i^* + \lambda_i) + \mathcal{O}(\zeta_i^2) \end{bmatrix}$$

$$\cong -2\begin{bmatrix} \zeta_i\omega_i T\lambda_i \\ j\sin(\omega_i T) \end{bmatrix} \tag{51}$$

Figure 1 shows the undamped discrete eigenvalues and denominator scalars ρ_{ij} and μ_{ij} in the complex plane. For small damping, the eigenvalues lie close to the unit circle, i.e., multiplied by the scalar α_i [see Eq. (35)]. Notice from Fig. 1 that if the system has distinct complex conjugate poles, the vectors λ_i and λ_j will never be collinear if $i \neq j$ so that $\rho_{ij} \neq 0$ and $\mu_{ij} \neq 0$. From Eqs. (50) and (51), note that only the denominator factor ρ_{ii} asymptotically approaches zero as the damping ratio approaches zero. Since a_i, a_j, b_i, and b_j are constants, the numerator factors $[Q_{ij}]_{11}$ and $[Q_{ij}]_{21}$ in Eq. (36) are also constants. This means that the diagonal block matrices of the grammian $[W_{c\infty}]_{ii}$, which contains the denominator factor ρ_{ii}, can be arbitrarily large as $\zeta_i \to 0$, whereas the magnitude of the off-diagonal block matrices $[W_{c\infty}]_{ij}$ is fixed. Thus the controllability grammian matrix for discrete flexible structures is diagonally dominant. Consider only the block diagonal terms, for $i = j$. Equations (37) and (38) simplify to

$$[Q_{ii}]_{11} = \lambda_i(a_i^2 + b_i^2) \cong 2\omega_i^2\lambda_i[1 - \cos(\omega_i T)] \tag{52}$$

$$[Q_{ii}]_{21} = \lambda_i(b_i + ja_i)^2 \cong 2\omega_i[1 - \cos(\omega_i T)] \tag{53}$$

Using Eqs. (52) and (53), one can simplify the block diagonal grammian in Eq. (36) to the form

$$\left[W_{c\infty}\right]_{ii} \cong \frac{\beta_{ii}^2[1 - \cos(\omega_i T)]}{2\omega_i^3}\begin{bmatrix} \dfrac{1}{\zeta_i T} & \dfrac{1}{\sin(\omega_i T)} \\ \dfrac{1}{\sin(\omega_i T)} & \dfrac{1}{\zeta_i T} \end{bmatrix} \tag{54}$$

Furthermore, only the diagonal elements of the block diagonal matrix are inversely proportional to the damping, so that the simplest approximation form can be written as follows.

Proposition 2:

$$\left[W_{c\infty}\right]_{ii} \cong \gamma_{ci}^2 I_{2\times 2} \tag{55}$$

where

$$\gamma_{ci}^2 = \frac{\beta_{ii}^2}{4\zeta_i\omega_i}\frac{2[1-\cos(\omega_i T)]}{\omega_i^2 T} \tag{56}$$

□

The first term in Eq. (56) corresponds to the ith controllability grammian for the corresponding continuous system. The term β_{ii} corresponds to the ith modal grammian for controllability.

Similarly for the state space parameterized as in Sec. II.B.2, the diagonal dominance of $[W_{c\infty}]_{ii}$ in Eq. (42) holds because it contains the denominator factor $\bar{\rho}_{ii}$, which can be arbitrarily large as $\zeta_i \to 0$. After some algebra, it follows that the block diagonal grammian in Eq. (42) can be expressed as

$$\left[W_{c_\infty}\right]_{ii} = \frac{1}{4\bar{\alpha}_i}\left\{\frac{a_{ii}+d_{ii}}{\delta_i T}I_{2\times 2} + \begin{bmatrix} a_{ii}-d_{ii} & -2b_{ii} \\ 2b_{ii} & -a_{ii}+d_{ii}\end{bmatrix} + \frac{1}{\tan(\psi_i T)}\begin{bmatrix} 2b_{ii} & a_{ii}-d_{ii} \\ a_{ii}-d_{ii} & 2b_{ii}\end{bmatrix}\right\} \tag{57}$$

Furthermore, only the first term in Eq. (57) is inversely proportional to damping so that the simplest form of the approximation can be written as follows.

Corollary 2:

$$\left[W_{c_\infty}\right]_{ii} \cong \frac{a_{ii}+d_{ii}}{4\delta_i T}I_{2\times 2} \tag{58}$$

□

IV. Observability Grammian

A. Definition

For the time interval (k_0T, k_1T), the discrete time observability grammian, $W_o(k_0, k_1)$, is defined by

$$W_o(k_0, k_1) = P_o^T(k_1-k_0)P_o(k_1-k_0) \tag{59}$$

where the discrete observability matrix is

$$P_o(k_1-k_0) = \begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \vdots \\ \tilde{C}\tilde{A}^{k_1-k_0-1}\end{bmatrix} \tag{60}$$

It can be shown that the preceding grammian satisfies

$$\tilde{A}^T W_o(k_0,k_1)\tilde{A} + \tilde{C}^T\tilde{C} = W_o(k_0,k_1) + \Phi(k_1,k_0)^T\tilde{C}^T\tilde{C}\Phi(k_1,k_0) \tag{61}$$

For asymptotically stable linear systems, the last term in Eq. (61) vanishes as $k_1 \to \infty$. This leads to the steady-state discrete time observability grammian W_{o_∞}, which satisfies the Sylvester equation

$$\tilde{A}^T W_{o_\infty}\tilde{A} + \tilde{C}^T\tilde{C} = W_{o_\infty} \tag{62}$$

B. Closed-Form Solution

Analogous to the controllability case, $(\tilde{A}, \tilde{B})$ can be replaced by $(\tilde{A}^T, \tilde{C}^T)$ so that a set of two-by-two Sylvester equations for the observability grammian satisfies

$$\tilde{A}_i^T\left[W_{o_\infty}\right]_{ij}\tilde{A}_j + [\tilde{C}^T\tilde{C}]_{ij} = \left[W_{o_\infty}\right]_{ij} \tag{63}$$

where $i, j = 1, \ldots, n$, and $[W_{o_\infty}]_{ij}$ is the (i, j)th two-by-two block of $[W_{o_\infty}]$. With the same approach as taken in Sec. III.B, it can be shown (see Appendix B) that the solution for the steady-state discrete time observability grammian for flexible structures is given as follows.

Proposition 3:

$$\left[W_{o_\infty}\right]_{ij} \cong -\frac{1}{2}\mathrm{Re}\left(\Xi_1^*\frac{[R_{ij}]_{22}}{\rho_{ij}} + \Xi_2^*\frac{[R_{ij}]_{12}}{\mu_{ij}}\right) \tag{64}$$

where

$$[R_{ij}]_{22} = \lambda_j\left\{\delta_{ij}^{11} + \delta_{ij}^{22} - j\left(\delta_{ij}^{21} - \delta_{ij}^{12}\right)\right\} \tag{65}$$

$$[R_{ij}]_{12} = \lambda_j\left\{-\delta_{ij}^{11} + \delta_{ij}^{22} - j\left(\delta_{ij}^{21} + \delta_{ij}^{12}\right)\right\} \tag{66}$$

□

For the state space parameterized as in Eqs. (15–17) analogous results hold. However, the output matrix appears in a different form. The outer product of the output matrix for the (i, j) block becomes

$$[\bar{C}^T\bar{C}]_{ij} = \bar{C}_{*i}^T\bar{C}_{*j} = \begin{bmatrix} \bar{\delta}_{ij}^{11} & \bar{\delta}_{ij}^{12} \\ \bar{\delta}_{ij}^{21} & \bar{\delta}_{ij}^{22}\end{bmatrix} \tag{67}$$

where

$$\bar{C}_{*i} = C_z[\mathrm{Re}(v_i), -\mathrm{Im}(v_i)] \tag{68}$$

$$\bar{\delta}_{ij}^{11} = \mathrm{Re}(v_i)^T C_z^T C_z \mathrm{Re}(v_j) \tag{69}$$

$$\bar{\delta}_{ij}^{12} = -\mathrm{Re}(v_i)^T C_z^T C_z \mathrm{Im}(v_j) \tag{70}$$

$$\bar{\delta}_{ij}^{21} = -\mathrm{Im}(v_i)^T C_z^T C_z \mathrm{Re}(v_j) \tag{71}$$

$$\bar{\delta}_{ij}^{22} = \mathrm{Im}(v_i)^T C_z^T C_z \mathrm{Im}(v_j) \tag{72}$$

Note the symmetry for $i = j$

$$\bar{\delta}_{ii}^{21} = \bar{\delta}_{ii}^{12} \tag{73}$$

This different form of the state and output matrix leads to the following result for the (i, j) block of the observability grammian.

Corollary 3:

$$\left[W_{o_\infty}\right]_{ij} = -\frac{1}{2\bar{\alpha}_j}\mathrm{Re}\left(\Xi_1^*\frac{[\bar{R}_{ij}]_{22}}{\bar{\rho}_{ij}} + \Xi_2^*\frac{[\bar{R}_{ij}]_{12}}{\bar{\mu}_{ij}}\right) \tag{74}$$

where

$$[\bar{R}_{ij}]_{22} = z_j\left\{\bar{\delta}_{ij}^{11} + \bar{\delta}_{ij}^{22} - j\left(\bar{\delta}_{ij}^{21} - \bar{\delta}_{ij}^{12}\right)\right\} \tag{75}$$

$$[\bar{R}_{ij}]_{12} = z_j\left\{-\bar{\delta}_{ij}^{11} + \bar{\delta}_{ij}^{22} - j\left(\bar{\delta}_{ij}^{21} + \bar{\delta}_{ij}^{12}\right)\right\} \tag{76}$$

□

Note that the preceding corollary is an exact relationship and is similar in form to the approximation in Proposition 3.

C. Diagonal Dominance

The diagonal dominance argument for the observability grammian is similar to the controllability case. From Eqs. (50) and (51), note that only the denominator factor ρ_{ii} asymptotically goes to zero as the damping ratio approaches zero. Since the terms δ_{ij}^{kl} are fixed constants, the numerator factors $[R_{ij}]_{22}$ and $[R_{ij}]_{12}$ in Eq. (64) will also be fixed constants. This means that the diagonal block matrices of the grammian $[W_{o_\infty}]_{ii}$, which contains the denominator factor ρ_{ii}, can be made arbitrarily large as $\zeta_i \to 0$, whereas the off-diagonal block matrices $[W_{o_\infty}]_{ij}$ will not. This represents the diagonal dominance property of the observability grammian for discrete flexible structures. Therefore, consider only the block diagonal terms. After some algebra, the block diagonal observability grammian in Eq. (64) can be reduced to the form

$$\left[W_{o_\infty}\right]_{ii} \cong \frac{1}{4}\left(\frac{\delta_{ii}^{11}+\delta_{ii}^{22}}{\zeta_i\omega_i T}I_{2\times 2} + \begin{bmatrix} \delta_{ii}^{11}-\delta_{ii}^{22} & 2\delta_{ii}^{12} \\ 2\delta_{ii}^{12} & -\delta_{ii}^{11}+\delta_{ii}^{22}\end{bmatrix} + \frac{1}{\tan(\omega_i T)}\begin{bmatrix} 2\delta_{ii}^{12} & -\delta_{ii}^{11}+\delta_{ii}^{22} \\ -\delta_{ii}^{11}+\delta_{ii}^{22} & -2\delta_{ii}^{12}\end{bmatrix}\right) \tag{77}$$

Furthermore, only the first term in Eq. (77) is inversely proportional to damping, so that the simplest form of the approximation can be written as follows.

Proposition 4:

$$\left[W_{o\infty}\right]_{ii} \cong \gamma_{oi}^2 I_{2\times 2} \tag{78}$$

$$\gamma_{oi}^2 = \frac{\left(\delta_{ii}^{11} + \delta_{ii}^{22}\right)}{4\zeta_i\omega_i T} = \frac{\hat{\theta}_i^2}{4\zeta_i\omega_i}\frac{1}{T} \tag{79}$$

□

The first term in Eq. (79) corresponds to the ith observability grammian for the corresponding continuous system.

Similarly for the state space parameterized as in Eqs. (15–17), the diagonal dominance of $[W_{o\infty}]_{ii}$ in Eq. (74) holds because it contains the denominator factor $\bar{\rho}_{ii}$, which can be arbitrarily large as $\zeta_i \to 0$. After some algebra, it follows that the block diagonal grammian in Eq. (74) can be expressed as

$$\left[W_{o\infty}\right]_{ii} = -\frac{1}{2\bar{\alpha}_i}\left(\frac{\bar{\delta}_{ii}^{11} + \bar{\delta}_{ii}^{22}}{\bar{\alpha}_i - \bar{\alpha}_i^{-1}} I_{2\times 2} + \frac{\bar{\alpha}_i}{\bar{\alpha}_i^4 - 2\bar{\alpha}_i^2\cos(2\psi_i T) + 1}\begin{bmatrix} -\Delta_1 & -\Delta_2 \\ -\Delta_2 & \Delta_1 \end{bmatrix}\right) \tag{80}$$

where

$$\Delta_1 = \left(-\bar{\delta}_{ii}^{11} + \bar{\delta}_{ii}^{22}\right)\left[\bar{\alpha}_i^2\cos(2\psi_i T) - 1\right] + 2\bar{\delta}_{ii}^{12}\bar{\alpha}_i^2\sin(2\psi_i T) \tag{81}$$

$$\Delta_2 = \left(-\bar{\delta}_{ii}^{11} + \bar{\delta}_{ii}^{22}\right)\bar{\alpha}_i^2\sin(2\psi_i T) - 2\bar{\delta}_{ii}^{12}\left[\bar{\alpha}_i^2\cos(2\psi_i T) - 1\right] \tag{82}$$

Furthermore, only the first term in Eq. (80) is inversely proportional to damping, so that the simplest form can be written as the following approximation.

Corollary 4:

$$\left[W_{o\infty}\right]_{ii} \cong \frac{\bar{\delta}_{ii}^{11} + \bar{\delta}_{ii}^{22}}{4\delta_i T} I_{2\times 2} \tag{83}$$

□

V. HSV

Because of the diagonal dominance property of the discrete controllability and observability grammian for flexible structures, the square of the ith HSV follows from Propositions 2 and 4.

Proposition 5:

$$\gamma_i^4 \cong \gamma_{oi}^2\gamma_{ci}^2 = \frac{1 - \cos(\omega_i T)}{8\omega_i^6\zeta_i^2 T^2} b_i b_i^T\left(c_{di}^T c_{di} + \omega_i^2 c_{ri}^T c_{ri}\right) \tag{84}$$

□

Similarly, for the state space parameterized as in Eqs. (15–17), Corollaries 2 and 4 lead to the approximate HSV.

Corollary 5:

$$\gamma_i^4 \cong \frac{(a_{ii} + d_{ii})\left(\bar{\delta}_{ii}^{11} + \bar{\delta}_{ii}^{22}\right)}{(4\delta_i T)^2} \tag{85}$$

□

Let the factors of deviations of the singular values of the discrete grammians from the singular values of the continuous grammians [as given by Eq. (6)] be defined by the following for the ith mode:

$$c_i = \frac{2[1 - \cos(\omega_i T)]}{\omega_i^2 T} \tag{86}$$

$$o_i = 1/T \tag{87}$$

In the limit when the sampling period approaches zero, the singular values of the scaled discrete grammians converge to continuous values, whereas the discrete HSV approaches the HSV of the continuous system[5–7] as follows.

Proposition 6:

$$\lim_{T\to 0} c_i\, 1/T = 1 \tag{88}$$

$$o_i\, T = 1 \tag{89}$$

$$\lim_{T\to 0} \gamma_i^4 = \left(\frac{\beta_{ii}\hat{\theta}_i}{4\zeta_i\omega_i}\right)^2 \tag{90}$$

where β_{ii}^2 and $\hat{\theta}_i^2$ are defined by Eqs. (7) and (8). □

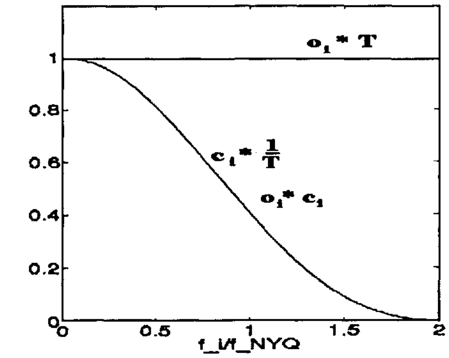
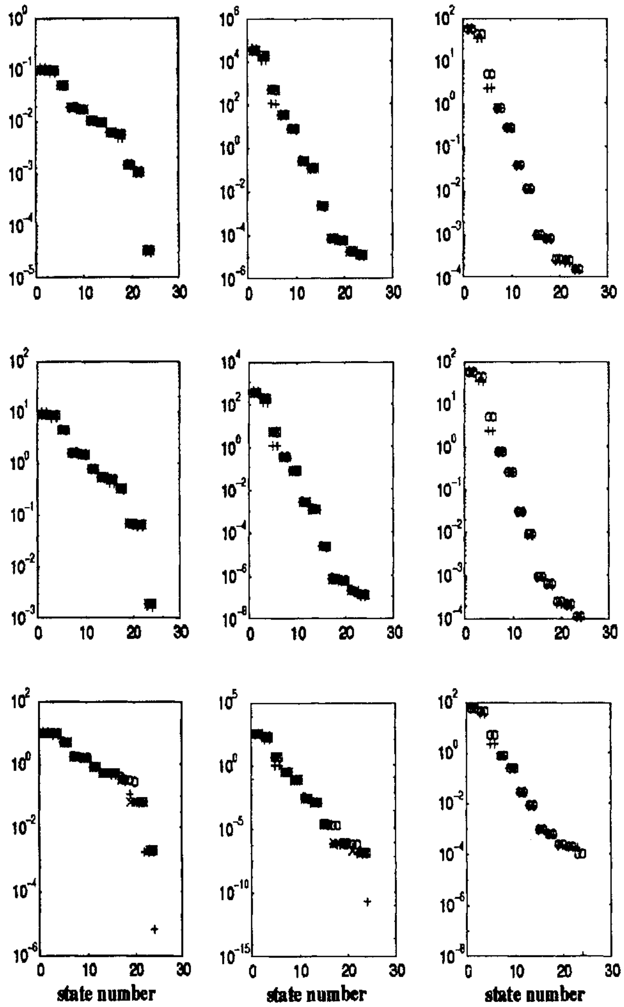


Fig. 2 Effect of sampling: deviations from continuous singular values.



$\sigma(W_c)$ — +, exact; ○, Eq. (54); and ×, Eq. (55)

$\sigma(W_o)$ — +, exact; ○, Eq. (77); and ×, Eq. (78)

HSV — +, exact and ○, Eq. (84)

Fig. 3 Exact and approximate singular values of grammians for CEM structure; $\zeta_i = 0.01$; first row: $q = 100$, second row: $q = 0.1$, and third row: $q = 0.0001$.

Note that without the sampling period scaling factor, the discrete controllability grammian approaches zero, whereas the discrete observability grammian approaches infinity. This result is consistent with the earlier and more general result involving principal component analysis (see Proposition 7 in Ref. 4). In addition, the preceding convergence of the discrete to continuous HSV for flexible structures is analogous to the more general result (see Proposition 8 in Ref. 4) where the singular values of the discrete Hankel matrix converges to the corresponding singular values of the grammians for the balanced system. For the state–space parameterized as in Eqs. (15–17), the HSV dependence on the inverse square of the sampling period

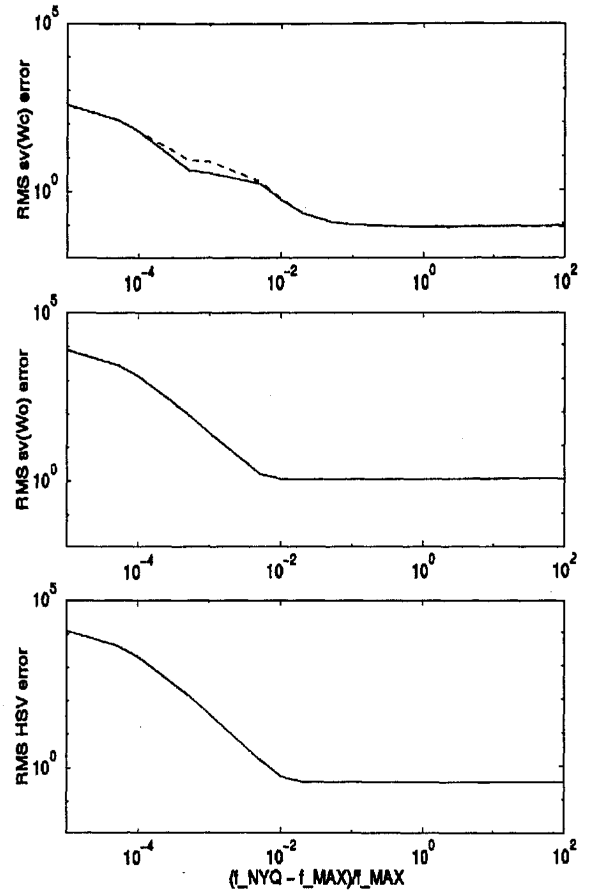


Fig. 4 The rms error in approximate singular values for CEM structure; ζ_i = 0.01; W_c error: ——, Eq. (54) and- - -, Eq. (55); W_o error: ——, Eq. (77) and - - -, Eq. (78); HSV error: ——, Eq. (84).

in Corollary 5 cancels with the numerator factor $(a_{ii} + d_{ii})$, which is proportional to the square of the sampling period as indicated by Eqs. (12) and (48). Indeed, similar results hold for the preceding type of parameterization in that the controllability and observability grammians go to zero and infinity, respectively, with decreasing sampling period.

The relationship between the discrete Hankel matrix $P_o P_c$ (Refs. 4 and 12) and the approximate formula for the HSV Γ^2 given in Eq. (84) is given next.

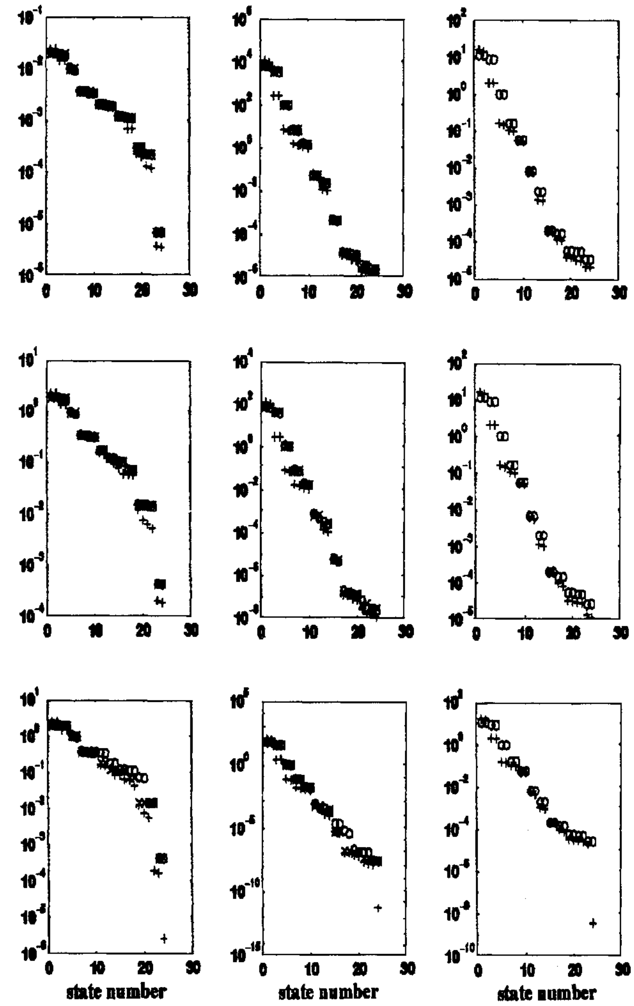
Proposition 7: Define the singular value decompositions $P_o = U_o \Sigma_o V_o^T$ and $P_c = U_c \Sigma_c V_c^T$, then

$$P_o P_c = R(W_o W_c)^{\frac{1}{2}} S^T \cong R\Gamma^2 S^T \tag{91}$$

where $R = U_o V_o^T$, $S = V_c U_c^T$, and $R^T R = I = S^T S$. □

Note that the singular values of the Hankel matrix is defined for an interval of time, whereas the analytical formula assumes a steady state or infinite time.

For comparison purposes with respect to the singular values of the continuous grammians, the factors c_i/T and $o_i T$ are used. This additional sampling period factor makes the singular values of the discrete grammian physically consistent with continuous singular values. Figure 2 shows the effect of sampling on the singular values of the observability and controllability grammians and the HSV as compared with the corresponding continuous singular values. At high sampling rates (for instance, $\omega_i/\omega_{\mathrm{NYQ}} \leq 0.2$), the predicted discrete singular values are close to the corresponding continuous singular values. Both the controllability and HSV decrease with slower sampling rate. The exact discrete singular values are expected to drop significantly in the neighborhood of Nyquist frequencies. This singularity near Nyquist is not predicted by the approximate analytical formula. In particular, the observability factor remains constant, which is counterintuitive, and hence this approximation appears to fail near the Nyquist frequency.



$\sigma(W_c)$	$\sigma(W_o)$	HSV
+, exact; ○, Eq. (54); and ×, Eq. (55)	+, exact; ○, Eq. (77); and ×, Eq. (78)	+, exact and ○, Eq. (84)

Fig. 5 Exact and approximate singular values of grammians for CEM structure; ζ_i = 0.05; first row: q = 100, second row: q = 0.1, and third row: q = 0.0001.

VI. Example

To validate the analytical formula, the exact and approximate grammians are computed for a former NASA experimental structure called the control-structures interaction evolutionary model (CEM) and is described in more detail in Refs. 8 and 9. A total of eight air thrusters are selected along with three displacement sensors. The structural model consists of $n_2 = 12$ modes whose first six modes are suspension modes. The frequencies are closely spaced and lightly damped, which is a typical phenomenon for this kind of structure. Case 1 assumes 1% damping ratio, whereas case 2 assumes 5% damping ratio for all modes. Note that a flexible structure with 5% damping ratios for all modes (case 2) will not usually be considered as lightly damped. This significant level of damping is used for the purpose of evaluating the level of the approximation errors in the singular value formulas.

Figure 3 shows the comparisons between the exact, Eq. (29), and the approximate singular values of the controllability, Eqs. (54) and (55), and observability, Eqs. (77) and (78), grammians and HSV, Eq. (84). The three rows of plots in Fig. 3 correspond to the sampling rates of

$$q = \frac{f_{\mathrm{NYQ}} - \max_i f_i}{\max_i f_i} = 100, 0.1, 0.0001 \tag{92}$$

where $\max_i f_i = \omega_{12}/2\pi$. The first two rows from Fig. 3 representing normalized sampling rates of $q = 100$ and 0.1 show that the

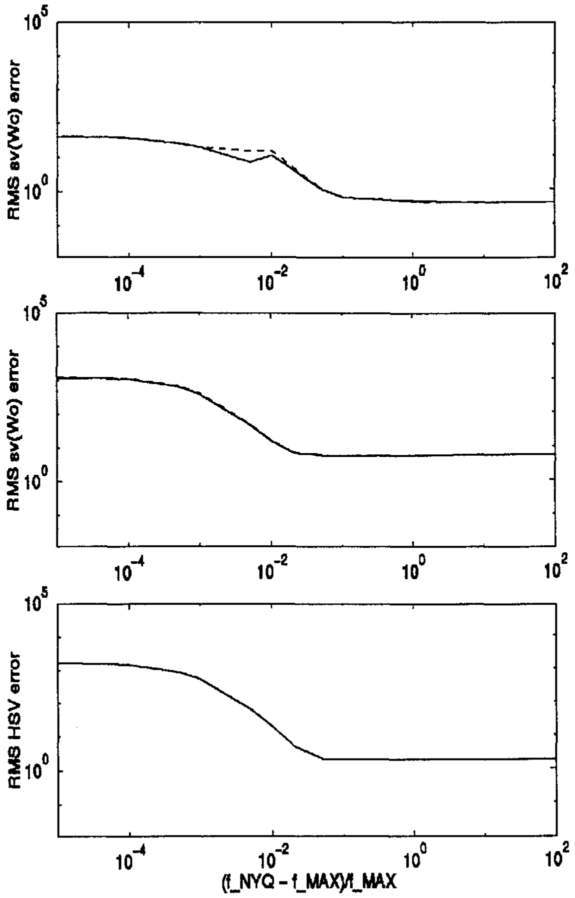


Fig. 6 The rms error in approximate singular values for CEM structure; $\zeta_i = 0.05$; W_c error: ——, Eq. (54) and - - -, Eq. (55); W_o error: ——, Eq. (77) and - - -, Eq. (78); HSV error: ——, Eq. (84).

approximate formula predicts the singular values accurately, up to frequencies near 90% of Nyquist frequency. However, the last row of plots ($q = 0.0001$) show a near singular condition represented by a large drop in the smallest singular value with increased errors in the remaining singular values. However, the last row corresponds to frequencies very close to Nyquist, i.e., $q = 0.0001$.

Figure 4 shows rms error plots of the approximate diagonal singular values for both types of approximations as a function of sampling rate $2 f_{\mathrm{NYQ}}$. Each error of the singular value is normalized by the corresponding exact value. The figure shows that the approximate formula predicts quite accurately down to Nyquist frequencies that are only 10% higher than the fastest mode. The normalized rms error is dominated by errors in the smallest singular values consistent with Fig. 3.

To evaluate the effect of larger damping ratios (case 2) in the approximate formulas for the singular values at different sampling frequencies, Fig. 5 shows the comparisons between the exact and the approximate singular values of the controllability and observability grammians and HSV. The three rows of plots in Fig. 5 correspond to the sampling rates in case 1. As in the lighter damping case, the approximate formula predicts the singular values accurately, up to frequencies near 90% of Nyquist frequency. The last row of plots similarly shows a near singular condition represented by a large drop in the smallest singular value with increased errors in the remaining singular values.

Figure 6 shows the approximate diagonal singular values as a function of sampling rate. Figure 6 shows that the approximate formula predicts quite consistently down to Nyquist frequencies that are only 10% higher than the fastest mode. The normalized rms error is again dominated by errors in the smallest singular values consistent with Fig. 5. The rms error significantly increases with the fivefold increase in damping ratios. However, it is noted that the damping ratios for case 2 are too large to be considered a lightly damped flexible structure.

VII. Conclusions

The results complement earlier work on continuous time flexible structure. For flexible structures modeled in discrete time, analytical expressions for singular values of controllability and observability grammian matrices and HSV are derived and validated through numerical examples. For the class of flexible structures with small damping and distinct frequencies, the preceding formulas are significantly simplified. It is found that the approximate formula is quite accurate up to near Nyquist frequencies. The discrete HSV converge to the approximate continuous formula with increased sampling rate. The simple but accurate approximate formula can provide useful physical insights in the selection of actuators and sensors, model reduction, and controller designs for flexible structures modeled in discrete time.

Appendix A: Proof of Proposition 1

By modal decomposition of the orthogonal two-by-two matrix in Eq. (21)

$$\Psi_i = X_i \Lambda_i X_i^H \tag{A1}$$

where

$$\Lambda_i = \mathrm{diag}(\lambda_i, \lambda_i^*) = \mathrm{diag}[\exp(j\omega_{di}T), \exp(-j\omega_{di}T)] \tag{A2}$$

$$X_i = \frac{1}{\sqrt{2}}\begin{bmatrix} j & -j \\ 1 & 1 \end{bmatrix} = X_j \tag{A3}$$

Eq. (34) can be decomposed, after premultiplying by X_i^H and postmultiplying by X_j, to obtain

$$\alpha_i \Lambda_i X_i^H [W_{c\infty}]_{ij} X_j - X_i^H [W_{c\infty}]_{ij} X_j \Lambda_j \alpha_j^{-1}$$
$$= -X_i^H [\tilde{B}\tilde{B}^T]_{ij} X_j \Lambda_j \alpha_j^{-1} \tag{A4}$$

It follows that the four elements of the two-by-two matrix $[W_{c\infty}]_{ij}$ satisfy

$$\begin{bmatrix} \rho_{ij}[\tilde{W}_{c\infty}]_{ij}^{11} & \mu_{ij}^*[\tilde{W}_{c\infty}]_{ij}^{12} \\ \mu_{ij}[\tilde{W}_{c\infty}]_{ij}^{21} & \rho_{ij}^*[\tilde{W}_{c\infty}]_{ij}^{22} \end{bmatrix} = Q_{ij} \tag{A5}$$

where

$$[W_{c\infty}]_{ij} = X_i [\tilde{W}_{c\infty}]_{ij} X_j^H \tag{A6}$$

$$[\tilde{W}_{c\infty}]_{ij} = \begin{bmatrix} [\tilde{W}_{c\infty}]_{ij}^{11} & [\tilde{W}_{c\infty}]_{ij}^{12} \\ [\tilde{W}_{c\infty}]_{ij}^{21} & [\tilde{W}_{c\infty}]_{ij}^{22} \end{bmatrix} \tag{A7}$$

$$Q_{ij} = -X_i^H [\tilde{B}\tilde{B}^T]_{ij} X_j \Lambda_j \alpha_j^{-1} \tag{A8}$$

and ρ_{ij} and μ_{ij} are defined by Eqs. (39) and (40). For small damping, Eq. (22) can be used to simplify the outer product $[\tilde{B}\tilde{B}^T]_{ij}$ appearing in Eq. (A8) to

$$[\tilde{B}\tilde{B}^T]_{ij} \cong M_i [BB^T]_{ij} M_j^T \tag{A9}$$

where $\tilde{M} = \text{blk-diag}(M_1, \ldots, M_n)$. Using the expression M_i in Eq. (23) and $[BB^T]_{ij}$, where

$$[BB^T]_{ij} = B_i B_j^T = \begin{bmatrix} \beta_{ij} & 0 \\ 0 & 0 \end{bmatrix} \tag{A10}$$

where β_{ij}^2 is defined by Eq. (7), the expression in Eq. (A9) can be expanded to

$$[\tilde{B}\tilde{B}^T]_{ij} \cong \frac{\beta_{ij}^2}{\omega_i^2 \omega_j^2} \begin{bmatrix} a_i a_j & a_i b_j \\ b_i a_j & b_i b_j \end{bmatrix} \tag{A11}$$

Using Eqs. (A2), (A3), and (A11), Q_{ij} in Eq. (A8) can be approximated as

$$Q_{ij} \cong -\frac{\beta_{ij}^2}{2\omega_i^2\omega_j^2}\begin{bmatrix}[Q_{ij}]_{11} & [Q_{ij}]_{21}^* \\ [Q_{ij}]_{21} & [Q_{ij}]_{11}^*\end{bmatrix} \tag{A12}$$

where $[Q_{ij}]_{11}$ and $[Q_{ij}]_{21}$ are defined in Eqs. (37) and (38). From Eq. (A5), the two-by-two matrix $[\tilde{W}_{c\infty}]_{ij}$ can be written as

$$\left[\tilde{W}_{c\infty}\right]_{ij} \cong -\frac{\beta_{ij}^2}{2\omega_i^2\omega_j^2}\begin{bmatrix}\frac{1}{\rho_{ij}}[Q_{ij}]_{11} & \frac{1}{\mu_{ij}^*}[Q_{ij}]_{21}^* \\ \frac{1}{\mu_{ij}}[Q_{ij}]_{21} & \frac{1}{\rho_{ij}^*}[Q_{ij}]_{11}^*\end{bmatrix} \tag{A13}$$

Finally, Eq. (36) is obtained from Eqs. (A6) and (A13).

Appendix B: Proof of Proposition 3

With the same approach as taken in the proof of Proposition 1, it can be shown that

$$\begin{bmatrix}\rho_{ij}^*\left[\tilde{W}_{o\infty}\right]_{ij}^{11} & \mu_{ij}\left[\tilde{W}_{o\infty}\right]_{ij}^{12} \\ \mu_{ij}^*\left[\tilde{W}_{o\infty}\right]_{ij}^{21} & \rho_{ij}\left[\tilde{W}_{o\infty}\right]_{ij}^{22}\end{bmatrix} = R_{ij} \tag{B1}$$

where

$$\left[W_{o\infty}\right]_{ij} = X_i\left[\tilde{W}_{o\infty}\right]_{ij}X_j^H \tag{B2}$$

$$\left[\tilde{W}_{o\infty}\right]_{ij} = \begin{bmatrix}\left[\tilde{W}_{o\infty}\right]_{ij}^{11} & \left[\tilde{W}_{o\infty}\right]_{ij}^{12} \\ \left[\tilde{W}_{o\infty}\right]_{ij}^{21} & [\tilde{W}_{o\infty}]_{ij}^{22}\end{bmatrix} \tag{B3}$$

$$R_{ij} = -X_i^H[\tilde{C}^T\tilde{C}]_{ij}X_j\Lambda_j^*\alpha_j^{-1} \tag{B4}$$

where ρ_{ij} and μ_{ij} are given in Eqs. (39) and (40). The output matrix product $[\tilde{C}^T\tilde{C}]_{ij}$ can be written as

$$[\tilde{C}^T\tilde{C}]_{ij} = C_{*i}^TC_{*j} = \begin{bmatrix}c_{ri}^Tc_{rj} & \frac{1}{\omega_j}c_{ri}^Tc_{dj} \\ \frac{1}{\omega_i}c_{di}^Tc_{rj} & \frac{1}{\omega_i\omega_j}c_{di}^Tc_{dj}\end{bmatrix} = \begin{bmatrix}\delta_{ij}^{11} & \delta_{ij}^{12} \\ \delta_{ij}^{21} & \delta_{ij}^{22}\end{bmatrix} \tag{B5}$$

Note that for $i = j$, $\delta_{ii}^{21} = \delta_{ii}^{12}$. For the special case of rate sensors only,

$$\delta_{ij}^{12} = \delta_{ij}^{21} = \delta_{ij}^{22} = 0; \qquad \delta_{ij}^{11} = c_{ri}^Tc_{rj} \tag{B6}$$

whereas for the case of displacement sensors only,

$$\delta_{ij}^{12} = \delta_{ij}^{21} = \delta_{ij}^{11} = 0; \qquad \delta_{ij}^{22} = \frac{c_{di}^Tc_{dj}}{\omega_i\omega_j} \tag{B7}$$

It can be shown that R_{ij} in Eq. (B4) can be approximated as

$$R_{ij} \cong -\frac{1}{2}\begin{bmatrix}[R_{ij}]_{22}^* & [R_{ij}]_{12} \\ [R_{ij}]_{12}^* & [R_{ij}]_{22}\end{bmatrix} \tag{B8}$$

where $[R_{ij}]_{22}$ and $[R_{ij}]_{12}$ are given by Eqs. (65) and (66). Finally, after some algebra, $[W_{o\infty}]_{ij}$ given by Eq. (64) is obtained.

Acknowledgment

The authors would like to thank Jer-Nan Juang of NASA Langley Research Center for discussions on the significance of the resulting formulas in system identification of flexible structures.

References

[1]Brockett, R. W., *Finite Dimensional Linear Systems*, Wiley, New York, 1970, Chap. 2.

[2]Kwakernaak, H., and Sivan, R., *Linear Optimal Control Systems*, Wiley, New York, 1972, Chap. 1.

[3]Skelton, R. E., *Dynamic Systems Control*, Wiley, New York, 1988, Chap. 5.

[4]Moore, B. C., "Principal Component Analysis in Linear Systems: Controllability, Observability, and Model Reduction," *IEEE Transactions on Automatic Control*, Vol. AC-26, No. 1, 1981, pp. 17–32.

[5]Gregory, C. Z., Jr., "Reduction of Large Flexible Spacecraft Models Using Internal Balancing Theory," *Journal of Guidance, Control, and Dynamics*, Vol. 7, No. 6, 1984, pp. 725–732.

[6]Williams, T., "Closed Form Grammians and Model Reduction for Flexible Space Structures," *IEEE Transactions on Automatic Control*, Vol. AC-35, March 1990, pp. 379–382.

[7]Gawronski, W., and Williams, T., "Model Reduction for Flexible Space Structures," *Journal of Guidance, Control, and Dynamics*, Vol. 14, No. 1, 1991, pp. 68–76.

[8]Lim, K. B., and Gawronski, W., "Actuator and Sensor Placement for Control of Flexible Structures," *Control and Dynamics Systems*, edited by C. T. Leondes, Vol. 57, Academic, San Diego, 1993, pp. 109–152.

[9]Gawronski, W., and Lim, K. B., "Balanced Actuator and Sensor Placement for Flexible Structures," AIAA Paper 95-3259, Aug. 1995.

[10]Gawronski, W., *Balanced Control of Flexible Structures*, Springer–Verlag, London, 1996.

[11]Franklin, G. F., Powell, J. D., and Workman, M. L., *Digital Control of Dynamic Systems*, Addison–Wesley, Reading, MA, 1992.

[12]Juang, J.-N., *Applied System Identification*, Prentice–Hall, Englewood Cliffs, NJ, 1994, Chap. 5.